ON THE REPRESENTATION OF EQUATIONS OCCURRING IN GEOMETRY, PHYSICS, ENGINEERING, AND OTHER FIELDS

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CONTENTS

INTRODUCTION

In section 1 this thesis attempts to describe simply and concisely some of the leading ideas upon which common nomograms, or graphical representations of equations, depend. In section 2 there is a brief description of nomographic literature with somewhat comprehensive indications of the types of available books on the subject.

Curved indices are mentioned in the literature but only a few simple cases are described. Although such indices may not be as practical as the straight indices which have been almost exclusively considered, their theoretical value seems to justify more consideration than they have so far received. In sections 3 - 7 we consider the possible use of conics as indices and add remarks concerning the use of other curves as indices.

Labeled indices are mentioned by D'Ocagne. In section 8 we discuss the notion of such indices in more detail and work out examples which seem simple enough to be practical. These examples provide a means for the graphic representation of the two equations

$$(1) \qquad f_3\sqrt{k^2 + (f_1 - f_2)^2} = kf_4$$

$$(2) \qquad f_1\,(f_1 + f_3) = f_4\sqrt{f_1{}^2 + f_2{}^2}$$

In section 9 we discuss combination of two straight line indices in such a way as to include the special cases usually described and to provide a means for devising new nomograms. Two nomograms which seem to be new are worked out.

In section 10 we consider the types of equations customarily met with. Five equation types, including nearly all practical cases, are selected and a nomogram for each is described. In the final section several ideas are mentioned which may assist in constructing alternative nomograms.

BASIC IDEAS

One of the more recent developments in the field of applied mathematics relates to the graphical representation of equations. Such representation is commonly comprised under the name "Nomography", and we shall use this term in that sense. A chart representing a particular equation will be called a nomogram. By its means we can readily find approximate solutions for the equation represented.

There are two types of nomogram in common use; Cartesian nomograms (or nomograms of intersection) and alignment nomograms (or index nomograms).

The simplest example of a Cartesian nomogram is the customary curve representation of an equation $f(x,y)=0$ in two variables. If we have an equation in three variables $f(x, y, z)=0$, we can make a Cartesian nomogram for it by the method of labeled contour lines. We fix one variable, draw the curve for the resulting equation of two variables, and label this curve with the value of the fixed variable. Choosing different values for the fixed variable, we thus obtain a system of labeled curves. This system together with the parallels to the x and y axes provides every point on the plane with three numbers, the two co-ordinates of the point and the label of the curve of the system passing through it. These three numbers satisfy the equation $f(x, y, z)=0$. Therefore, when two of the variables x, y, z are known, the third can be read off from the chart.

Cartesian nomograms are, of course, seldom used for equations involving more than three variables because of the difficulty in constructing and using more than three networks of curves on one sheet of paper.

Often the construction of these nomograms is tedious. By making use of functional scales on one or both of the co-ordinate axes it may be possible to transform the curves into straight lines which are not only easier to draw but give more accurate results. A functional scale is constructed as in the following example: Consider the function log t. Choose an origin and a positive direction. Place the label t at the distance log t from the origin. The resulting scale is the logarithmic scale with t as silent coordinate.

Suppose we wish to construct a nomogram for $x^2 \cdot y^3 = z$. If one of the variables is fixed, the equation represents a curve which isn't straight but if we take logarithms we get $2\log x + 3\log y = \log z$. If we choose $x' = 2\log x$ as the scale to be carried on the x axis and $y' = 3\log y$ as that to be carried on the y axis, our equation becomes $x' + y' = \log z$ which represents a straight line for fixed z.

An alignment nomogram for an equation $f(u, v, w)=0$ is a system of three curves *labeled respectively with the values of the three variables

*(such curves are called supports)

u, v, and w such that any straight line intersecting these curves will determine values of the variables satisfying the given equation. These nomograms cannot represent all the equations in three variables but only those of certain forms. Very frequently, however, equations requiring representation have precisely such forms. Their representation by alignment nomograms is often simpler in construction and more convenient in use than their Cartesian nomograms. Two or more alignment nomograms may be combined to furnish nomograms for four or more variables.

Alignment nomograms have been pre-eminently developed by Maurice D'Ocagne since 1884. Their theory grows out of the condition for the collinearity of three points (x_1, y_1), (x_2, y_2), (x_3, y_3):

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Assume that we have a nomogram for an equation involving the three variables u, v, and w. Our three equations are then given by the three pairs of parametric equations:

$$(1) \quad x = f_1(u) \quad , \quad y = g_1(u)$$

$$(2) \quad x = f_2(v) \quad , \quad y = g_2(v)$$

$$(3) \quad x = f_3(w) \quad , \quad y = g_3(w)$$

The equation represented by our nomogram is therefore

$$\begin{vmatrix} f_1(u) & g_1(u) & 1 \\ f_2(v) & g_2(v) & 1 \\ f_3(w) & g_3(w) & 1 \end{vmatrix} = 0$$

If f_1, f_2, and f_3 are constants, our curves reduce to parallel straight lines. Similarly, other special alignment nomograms are obtained when other particular conditions are imposed upon the functions f_ν, g_ν.

If f_1 and g_1 are both functions of the same two variables u and t then the first curve is replaced by a family of labeled curves each member of the family being determined when a particular value is assigned to t. Thus we get an alignment chart involving four variables. If each of the original three curves is replaced by a family of labeled curves we have a nomogram for an equation in six variables.

By using the rules for manipulating the elements of a determinant without changing its value we have opportunity to secure different nomograms for the same equation.

If an equation involves a great many variables it can be reduced to a series of equations with fewer variables -- if we wish, no more than three. Then, if each of these latter can be represented, the original equation can be represented nomographically. For example, $x + y + z + t = 0$ can be reduced to the two equations $x + y = p$ and $p + z + t = 0$

LITERATURE

Nomography has been developed for the most part by French mathematicians. An original and comprehensive work is the "Traité de Nomographie" of D'Ocagne. In this treatise is included a general theory which covers a large class of nomograms. Many authors specify that their work is merely an adaptation of the methods of D'Ocagne. Few have introduced new ideas. Among these few are Soreau, Goedseels, and Clark.

Nomography is a valuable aid to the engineer and other applied scientists. It seems that the literature may be divided into two parts. One of these is written by mathematicians in a language not generally understood by men in applied work, and the other by the men who use nomography but do not take advantage of the basal mathematical notions involved or of the opportunity for flexibility and range of application offered by generalized mathematical considerations.

Many of the more recent books have titles such as "Construction of Nomograms". These are intended to give construction methods with little or no reference to the ideas involved. This is regrettable since the ideas are often simpler than the unexplained detail. Then, too, a particular equation cannot be dealt with satisfactorily -- if its representation is at all important -- unless the well known methods, at least, are systematically tried on it.

Many times a person not especially interested in the study of nomography discovers that a nomogram for some particular equation would aid him greatly in his work. If he wishes to construct this nomogram he will find "Graphical and Mechanical Computation" of Lipka a valuable reference. Equations are classified, only simple mathematical notions are used, and examples are worked out.

If a person becomes interested in the theory of nomography he will find other books more interesting. "A First Course in Nomography" by S. Brodetsky is written in simple language and introduces the fundamentals of the subject. The book of Frechet and Roullet, who are mathematician and engineer respectively, also gives a clear introduction to the mathematical ideas. Here little attention is paid to the details of any particular nomogram.

Two books which go into greater detail are "The Design of Diagrams for Engineering Formulas" by Hewes and Seward and "The Nomogram" by Alcock and Jones. Both books are written in a practical manner and emphasize the use of the determinant.

A concise, theoretical, and practical study is "A Short Monograph on

Nomography" by F. M. Wood. This is a reprint from The Engineering Journal, June and August 1930.

In the past ten years an increasingly large number of articles has been appearing in periodicals concerning nomography and its application. Most of these are merely detailed descriptions of some simple nomogram. Those interested will find many articles listed in the various periodical indices. Especially numerous are those listed in The Industrial Arts Index under the headings of "Nomography" and "Charts (calculating)".

CURVED INDICES

The type of index line has much to do with the nature of a nomogram. The straight line has been generally adopted as the most practical index. It should be noted that, so far, equations met with for which a nomogram has been desired are not of marked diversity of type. However, theoretical mathematical work should be far in advance of practical needs.

Alignment nomograms may be extended to include those with curved indices. There seems to be little discussion of such indices in the literature. M. Goedseels considers two parallel straight lines and other arrangements of two straight lines as indices thus devising nomograms for relations between four variables. He has also considered circles of fixed radius. M. N. Gercevanoff has used concentric circle index lines.

The idea of using various types of indices has been generally dismissed with the statement that they would be of only theoretical value. This statement is questionable. It would be difficult to show conclusively that a suggested method can have no value in practical work. The choice of nomogram involves many considerations such as degree of accuracy, range of the variables, size of the chart, and ease of obtaining results. Theoretical possibilities that at first seem uninviting may, upon further examination, turn out to be practically servicable.

Difficulty in constructing supports is not a serious draw-back. Usefulness may justify laborious preparation of a nomogram.

It would appear that nomograms with curves as indices have been deemed impractical because of apparent difficulties in constructing the variable indices. However, some curves not straight may be easily constructed by graphical methods. For example, a parabola can be drawn graphically by integrating a straight line, the representation of a cubic function may be secured by integrating a quadratic and so on. Various curves may be constructed by means of a number of given points. Graphical integration and other graphical processes as given in "Graphical Methods" of Runge, other books on graphical methods, and books on projective geometry are not too cumbersome.

We may for example make use of the fact that a conic may be constructed from five given points by Pascal's theorem. With the conic as an index we may represent nomographically an equation involving six variables, for, if five points are fixed upon five respective supports, we can draw the part of the conic which intersects the sixth support and thus find the values of the sixth variable. If the sixth support is a straight line, a special method of finding its intersection with the conic is available from the known methods of projective geometry.

In view of these facts it does not seem justified to discard curved index lines.

THE VERTICAL PARABOLA AS AN INDEX

We shall now determine the equation represented by a nomogram consisting of four fixed supports and a variable index curve which is a vertical parabola i.e. one having its axis parallel to the y axis of our assumed Cartesian coordinate system. There will then be four variables carried upon the four respective supports. If values of three of the variables are given, three points are determined upon the corresponding supports. These three points will determine a vertical parabola which will determine the value or values of the fourth variable where it intersects the fourth support. The equation of the parabola is

$$Ax^2 + Bx + Cy + D = 0$$

A necessary and sufficient condition that four points (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) lie upon the parabola is:

$$\begin{vmatrix} x^2_1 & x_1 & y_1 & 1 \\ x^2_2 & x_2 & y_2 & 1 \\ x^2_3 & x_3 & y_3 & 1 \\ x^2_4 & x_4 & y_4 & 1 \end{vmatrix} = 0$$

The four fixed supports are then given by the four pairs of parametric equations

$$(1) \quad x = f_1(u) \quad , \quad y = g_1(u)$$

$$(2) \quad x = f_2(v) \quad , \quad y = g_2(v)$$

$$(3) \quad x = f_3(w) \quad , \quad y = g_3(w)$$

$$(4) \quad x = f_4(t) \quad , \quad y = g_4(t)$$

These supports are constructed point by point, each point being labeled with the respective value of u, v, w, or t which determines it. If four points lie upon a vertical parabola they will determine values of the variables satisfying the equation:

$$\begin{vmatrix} f^2_1 & f_1 & g_1 & 1 \\ f^2_2 & f_2 & g_2 & 1 \\ f^2_3 & f_3 & g_3 & 1 \\ f^2_4 & f_4 & g_4 & 1 \end{vmatrix} = 0$$

This is the general equation which may be represented nomographically by four fixed supports and a vertical parabola as index.

If the four supports are straight lines all parallel to the y axis, f_u are constants. We choose the intersections of the supports with the x axis as the origins of the respective scales. We may take $f_1 = 0$ without loss of generality. Let $f_2 = k$, $f_3 = L$, and $f_4 = m$. The equation represented is then

$$\begin{vmatrix} 0 & 0 & g_1 & 1 \\ k^2 & k & g_2 & 1 \\ L^2 & L & g_3 & 1 \\ m^2 & m & g_4 & 1 \end{vmatrix} = 0$$

or $\quad g_1' + g_2' + g_3' + g_4' = 0$

This equation can, of course, be more conveniently represented by means of straight index lines.

Other equations representable by this kind of nomogram are:

(1) $$\frac{f_1\, f_4}{f_2\, f_3} + \frac{f_1 - f_4}{f_2 - f_3 - f_4} = 0$$

(2) $$\frac{f_1\, f_2}{f_3\, f_4} + \frac{f_1 - f_2 - f_4}{f_3 + k} = 0$$

<u>OTHER PARABOLAS AS INDICES</u>

We might drop the condition that the parabola be vertical. Thus it would take four points to determine the curve. With this index we could devise nomograms for equations in five variables.

We could require that the vertical parabola fulfill another condition such as passing through a fixed point or being tangent to the y axis. Such a condition would reduce the nomogram to one of three variables. Theoretically,